

Micropolar fluid system in a space of distributions and large time behavior

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Abstract

We analyze the well-posedness of the initial value problem for the generalized micropolar fluid system in a space of tempered distributions and also prove the existence of the stationary solutions. The asymptotic stability of solutions is showed in this space, and as a consequence, a criterium for vanishing small perturbations of initial data (stationary solution) at large time is obtained. A fast decay of the solutions is obtained when we assume more regularity on the initial data.

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1. Introduction

In this paper we study the well-posedness of the generalized micropolar system in \mathbb{R}^3 (3DGMP) in a tempered distributions spaces and analyze the asymptotic stability of solutions in these spaces. This model is given by the following equations system

$$\frac{\partial u}{\partial t} + (\chi + \nu)(-\Delta)^{\gamma} u + u \cdot \nabla u + \nabla \pi - 2\chi \nabla \times \omega = f, \quad x \in \mathbb{R}^3, \quad t > 0, \quad (1)$$

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$$\operatorname{div} u = 0, \quad x \in \mathbb{R}^3, \quad t > 0, \quad (2)$$

$$\frac{\partial \omega}{\partial t} + \mu(-\Delta)^\beta \omega + u \cdot \nabla \omega + 4\chi \omega - \kappa \nabla \operatorname{div} \omega - 2\chi \nabla \times u = g, \quad x \in \mathbb{R}^3, \quad t > 0, \quad (3)$$

$$u(0, x) = u_0(x), \quad \omega(0, x) = \omega_0(x), \quad x \in \mathbb{R}^3, \quad (4)$$

with $\beta, \gamma > 0$. The unknowns are $u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$, $\omega(x, t) = (\omega_1(x, t), \omega_2(x, t), \omega_3(x, t)) \in \mathbb{R}^3$ and $\pi(x, t) \in \mathbb{R}$, representing, respectively, the linear velocity and the velocity field of rotation of the fluid particles and the pressure of fluid. The fields f and g are external forces and moments, respectively. The constants $\kappa = (c_0 + c_d - c_a)$, $\chi, \nu, \mu = c_a + c_d$ represent viscosity coefficients, which determine fluid physical characteristics, in particular, χ is the Newtonian viscosity and ν, c_0, c_a, c_d , are new viscosities related to the structure of the external rotation field ω . These constants satisfy $c_0 + c_d > c_a$. It is assumed that the density of the fluid is equal to one. u_0 and ω_0 represent the initial velocities and we assume that $\operatorname{div} u_0 = 0$. When $\gamma = \beta = 1$, we obtain the standard micropolar system developed by A.C. Eringen [2] that enables us to consider some physical phenomena which cannot be analyzed using the classical Navier–Stokes model for the incompressible fluid. In fact, the micropolar fluid model are non-Newtonian fluids with nonsymmetric stress tensor and it may represent fluids consisting of randomly oriented particles suspended in a viscous medium, when the deformation of fluid particles is ignored. Observe that if the microrotation viscosity χ equals zero, the system (1)–(4) reduces to the Navier–Stokes system, describing a purely viscous fluid, which has been greatly analyzed, and the velocity field is independent of the microrotation field (see for instance, the classical books by Ladyzhenskaya [4], Temam [10]).

We use the standard notations ∇ , Δ , div and $\nabla \times$ for the gradient, Laplacian, divergence and rotational operators, respectively. The i th component of $(u \cdot \nabla)u$ and $(u \cdot \nabla)\omega$ in Cartesian coordinates is given, respectively, by

$$[(u \cdot \nabla)u]_i = \sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j}; \quad [(u \cdot \nabla)\omega]_i = \sum_{j=1}^3 u_j \frac{\partial \omega_i}{\partial x_j}.$$

Several papers are devoted to the existence and uniqueness of solutions to stationary and nonstationary problems for micropolar fluids system (1)–(4) (in the case $\beta = \gamma = 1$); see for instance [7] and the references therein and [8,9] for a coupled system of the micropolar fluid (1)–(4) with the Maxwell's equations and Ohm's law. In the last decades, the study of singularities, global existence and long time behavior for some models in fluid mechanics have become a holy grail in the mathematical community. However, some results are not known from the point of view of existence, uniqueness, asymptotic stability, behavior at infinity for the system (1)–(4) neither cases $\beta = \gamma = 1$ nor $\beta = \gamma \neq 1$ in the space of temperate distributions which we will denote by PM^a -spaces whose precise definition will be given later.

New aspects of the study on the micropolar fluid problems are considered in this paper. In fact, firstly our aim is to show the existence and uniqueness of mild solutions, asymptotic stability results for the micropolar system (1)–(4) at PM^a -spaces and, as a consequence, a criterium for vanishing small perturbations of initial data at large time is obtained. Moreover, we will show that if the initial data lies on the closure of the intersection of two PM^a -spaces, the solution vanishes when time is large. Our approach to study global solutions to system (1)–(4), as well as the large time behavior, can be applied to the case of stationary solutions, and in this way we give an integral formulation satisfied by stationary solutions and describe some consequences of the results of well-posedness in the PM^a -spaces.

From another point of view, we show that PM^a -spaces allow the existence of initially-singular solutions. These solutions are instantaneously smoothed out if initially they are small enough. However, we show the existence of global solutions at PM^a -space with $a = 4 - 2\gamma$ for which we do not know whether or not the singularity persists.

The existence of solutions for the Navier–Stokes equations has been treated at the space of tempered distributions PM^a in [1]. In the same work, the authors study the asymptotic stability of small solutions.

Without loss of generality, from this line onwards, we will take $\kappa = \mu = 1$ and $\nu = \chi = \frac{1}{2}$. The basic properties of PM^a -spaces will be reviewed in the next section. Section 3 will be devoted to show well-posedness theorems and existence of stationary solutions. The decay rates when we take more regular initial data will be discussed in Section 4. Finally, in Section 5 we analyze the asymptotic stability of solutions and we give a criterium of vanishing small perturbation of initial data at large time.

2. Function spaces and definitions

In this section we introduce the relevant functional spaces to our study of solutions regarding the Cauchy problem for system (1)–(4). We list some facts about convolution and we discuss the notion of solution at these spaces.

2.1. PM^a -space and the Leray projector

We define the space PM^a as

$$PM^a \equiv \left\{ v \in \mathcal{S}' : \hat{v} \in L^1_{\text{loc}}(\mathbb{R}^n), \|v\|_a \equiv \operatorname{ess\,sup}_{\xi \in \mathbb{R}^n} |\xi|^a |\hat{v}(\xi)| < \infty \right\},$$

where $a \in \mathbb{R}$ is a given parameter.

The Riesz potential operator $(-\Delta)^r$ is defined as usual through the Fourier transform as

$$((-\Delta)^r f)^\wedge(\xi) = |\xi|^{2r} \hat{f}(\xi),$$

with the convention

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx.$$

The lemma below deals with the continuity of Riesz transform at PM^a -spaces. We recall that the Leray projector on solenoidal vector fields is given by

$$\mathbb{P}u = u - \nabla \Delta^{-1}(\nabla \cdot u),$$

for sufficiently smooth field $u = (u_1(x), u_2(x), u_3(x))$. Moreover, we recall that \mathbb{P} is a matrix $n \times n$ with elements

$$\mathbb{P}_{k,j} = \delta_{kj} + R_k R_j,$$

where R_j ($j = 1, 2, 3$) are the Riesz transforms which are pseudodifferential operators defined in the Fourier variable as $\widehat{R_j f}(\xi) = \frac{i\xi_j}{|\xi|} \hat{f}(\xi)$. Furthermore $\widehat{\mathbb{P}}(\xi)$ is a matrix $n \times n$ with elements

$$\widehat{\mathbb{P}_{k,j}}(\xi) = \delta_{kj} - \frac{\xi_i \xi_j}{|\xi|^2}.$$

In this way, in order to prove the continuity of Leray operator \mathbb{P} on the PM^a -spaces, it is sufficient to prove the continuity of Riesz transform R_j , $j = 1, \dots, n$. We prove the following lemma.

Lemma 2.1. *The Riesz transform $R_j = \partial_j(-\Delta)^{-\frac{1}{2}}$, $j = 1, 2, \dots, n$, is continuous at PM^a -spaces, $a \in \mathbb{R}$.*

Proof. The Riesz transform is a bounded operator in PM^a , because

$$\begin{aligned} \|R_j v\|_a &= \operatorname{ess\,sup}_{\xi \in \mathbb{R}^n} |\xi|^a |\widehat{R_j v}(\xi)| = \operatorname{ess\,sup}_{\xi \in \mathbb{R}^n} |\xi|^a \left| \frac{\xi_j}{|\xi|} \hat{v}(\xi) \right| \\ &\leq \operatorname{ess\,sup}_{\xi \in \mathbb{R}^n} |\xi|^a |\hat{v}(\xi)| = \|v\|_a, \end{aligned} \quad (5)$$

which proves the lemma. \square

Now, we recall a fact about convolution that will be useful to perform some estimates at PM^a -spaces:

Proposition 2.2 (Convolution of singular kernels [6]). *Let $0 < \alpha < n$, $0 < \beta < n$ and $0 < \alpha + \beta < n$. Then we have*

$$(|x|^{\alpha-n} * |x|^{\beta-n})(y) = \int_{\mathbb{R}^n} |z|^{\alpha-n} |y-z|^{\beta-n} dz = C(\alpha, \beta, n) |y|^{\alpha+\beta-n}. \quad (6)$$

3. Well-posedness in PM^a

The aim of this section is to describe the results of existence, uniqueness and continuity of solutions for the system (1)–(4) with respect to initial data at PM^a -spaces.

We start with the definition of time dependent functional spaces required for the study of the initial value problem (1)–(4). Spaces of scalar-value and vector-value distributions will be abusively denoted in same way. If X represents a Banach space, we will denote the set of all functions $u \in C((t_1, t_2); X)$ such that $\sup_{t_1 < t < t_2} \|u(t)\|_X < \infty$, by $BC((t_1, t_2); X)$.

Definition 3.1. Let us define for all $a \geq 0$ and $0 < T \leq \infty$, $F_{a,T}$ the space of all time-dependent distributions $h(t, x) \in BC((0, T); PM^a)$ with norm

$$\|h\|_{F_{a,T}} = \sup_{0 < t < T} \|h(t)\|_a,$$

which are weakly continuous in the distributions sense at $t = 0$, i.e., there exists $h_0 \in PM^a$ such that

$$|\langle h(t) - h_0, \varphi \rangle| \rightarrow 0$$

when $t \rightarrow 0^+$ for all φ in the Schwartz space \mathcal{S} .

If $T = \infty$, we denote $F_{a,\infty} \equiv F_a$. Let us also define $F_{q,a}$ by

$$F_{q,a} = \left\{ h \in F_a : t^{\frac{\alpha}{2}} h(t, x) \in BC((0, \infty); PM^a) \right\} \quad \text{where } \alpha = \frac{q-a}{\gamma}, \quad \gamma > 0.$$

The norm in $F_{q,a}$ is naturally defined by

$$\|\cdot\|_{F_{q,a}} = \sup_{t>0} t^{\frac{\alpha}{2}} \|\cdot\|_q + \sup_{t>0} \|\cdot\|_a.$$

3.1. The linearized system and mild solutions

For exposition simplicity we will take momentarily, external sources of particle linear and angular momentum as equal to zero, and $\chi = \nu = \frac{1}{2}$ and $\kappa = \mu = 1$. For general cases, our arguments remain valid with a slight modification. Hence, applying the Leray projector to Eq. (1), the system (1)–(4) is expressed as

$$\frac{\partial u}{\partial t} + (-\Delta)^\gamma u - \nabla \times \omega + \mathbb{P}(u \cdot \nabla u) = 0, \quad (7)$$

$$\frac{\partial \omega}{\partial t} + (-\Delta)^\beta \omega + u \cdot \nabla \omega + 2\omega - \nabla \operatorname{div} \omega - \nabla \times u = 0, \quad (8)$$

$$u(0, x) = u_0(x), \quad \omega(0, x) = \omega_0(x). \quad (9)$$

We recall that applying the divergence operator to Eq. (1), as usual, we obtain the term

$$\Delta \pi = -\operatorname{div}(u \cdot \nabla u),$$

and therefore the pressure force π may be recovered by

$$\nabla \pi = -\nabla \Delta^{-1} \operatorname{div}(u \cdot \nabla u) = (\mathbb{P} - I)(u \cdot \nabla u).$$

Now, consider the linearized system associated to system (7)–(9)

$$\frac{\partial u}{\partial t} + (-\Delta)^\gamma u = \nabla \times \omega, \quad x \in \mathbb{R}^3, \quad t > 0, \quad (10)$$

$$\frac{\partial \omega}{\partial t} + (-\Delta)^\beta \omega + 2\omega - \nabla \operatorname{div} \omega = \nabla \times u, \quad x \in \mathbb{R}^3, \quad t > 0. \quad (11)$$

Applying the Fourier transform to the system (10)–(11), we obtain

$$\frac{\partial \hat{u}}{\partial t} + |\xi|^{2\gamma} \hat{u} = \widehat{(\nabla \times \omega)}, \quad x \in \mathbb{R}^3, \quad t > 0, \quad (12)$$

$$\frac{\partial \hat{\omega}}{\partial t} + |\xi|^{2\beta} \hat{\omega} + 2\hat{\omega} - \widehat{(\nabla \operatorname{div} \omega)} = \widehat{(\nabla \times u)}, \quad x \in \mathbb{R}^3, \quad t > 0. \quad (13)$$

Using the notation $y = (u, \omega)$, we can write the system (12)–(13) in a more compact form as

$$\frac{\partial \hat{y}}{\partial t} + A(\xi) \hat{y} = 0,$$

where

$$A(\xi) = \begin{bmatrix} |\xi|^{2\gamma} I & B(\xi) \\ B(\xi) & R(\xi) + (|\xi|^{2\beta} + 2)I \end{bmatrix} = A_1(\xi) + A_2(\xi) + A_3(\xi), \quad (14)$$

$$A_1(\xi) = \begin{bmatrix} |\xi|^{2\gamma} I & 0 \\ 0 & (|\xi|^{2\beta} + 2)I \end{bmatrix}, \quad A_2(\xi) = \begin{bmatrix} 0 & B(\xi) \\ B(\xi) & 0 \end{bmatrix},$$

$$A_3(\xi) = \begin{bmatrix} 0 & 0 \\ 0 & R(\xi) \end{bmatrix},$$

with

$$R(\xi) = \begin{bmatrix} \xi_1^2 & \xi_1 \xi_2 & \xi_1 \xi_3 \\ \xi_1 \xi_2 & \xi_2^2 & \xi_2 \xi_3 \\ \xi_1 \xi_3 & \xi_2 \xi_3 & \xi_3^2 \end{bmatrix} \quad \text{and} \quad B(\xi) = i \begin{bmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{bmatrix}.$$

Note that $A_1(\xi)$, $A_2(\xi)$ and $A_3(\xi)$ are Hermitian matrices.

Let $y_0 = (u_0, \omega_0) \in PM^a$. For each $t \geq 0$, let us define $G(t)$ as the operator at PM^a such that

$$(\widehat{G(t)y_0})(\xi) = e^{-A(\xi)t} \hat{y}_0,$$

where $A(\xi)$ has been defined in (14). Equivalently, the matrix $(e^{-A(\xi)t})^\vee$ is the fundamental solution of the linear system (10)–(11). Note also that $\{G(t)\}_{t \geq 0}$ is a semi-group on PM^a -spaces.

3.1.1. Eigenvalues of $A(\xi)$ and semi-group estimates

From now on, given a matrix M , let us denote by $\sigma(M)$ the set of eigenvalues of M . It is not difficult to prove that $\sigma(R(\xi)) = \{0, 0, |\xi|^2\}$. Indeed, note that all minors of $R(\xi)$ are zero and consequently the characteristic polynomial must be

$$P(\lambda) = \lambda^3 - (\text{Tr}(R(\xi)))\lambda^2 = \lambda^3 - |\xi|^2\lambda^2,$$

which has roots $\{0, 0, |\xi|^2\}$.

A simple computation shows that $\sigma(B(\xi)) = \{-|\xi|, 0, |\xi|\}$ with respective eigenvectors $\{v_1, v_2, v_3\}$ which are an orthogonal base in \mathbb{C}^3 . Consequently, it is easy to show that $\sigma(A_2(\xi)) = \{-|\xi|, -|\xi|, 0, 0, |\xi|, |\xi|\}$. Hence, we have the decomposition

$$\mathbb{C}^6 = V_1 \oplus V_2 \oplus V_3, \quad (15)$$

with

$$\begin{aligned} V_1 &= \text{Ker}(A_2(\xi) + |\xi|I) = \text{Span}\langle (v_1, v_1), (v_3, -v_3) \rangle, \\ V_2 &= \text{Ker}(A_2(\xi)) = \text{Span}\langle (v_2, v_2), (v_2, -v_2) \rangle, \\ V_3 &= \text{Ker}(A_2(\xi) - |\xi|I) = \text{Span}\langle (v_3, v_3), (v_1, -v_1) \rangle. \end{aligned} \quad (16)$$

Since $A_2(\xi)$ is a Hermitian matrix, we can consider the base $\{(v_i, \pm v_i)\}_{i=1}^3$ as an orthonormal base in \mathbb{C}^6 .

For $\xi \neq 0$, note that $A_1(\xi)$ is positive definite (all eigenvalues are positive) and $A_3(\xi)$ is semipositive definite (because $R(\xi)$ has also this property), i.e.,

$$\bar{v}A_3(\xi)v \geq 0, \quad \text{for all } v \in \mathbb{C}^3.$$

Note that if $h_i \perp d_i$ in \mathbb{C}^3 for $i = 1, 2$, then

$$(\overline{h_1}, \overline{h_2})A_1(\xi)(d_1, d_2) = 0.$$

Now, let $w \in \mathbb{C}^6$. By the orthogonal decomposition (15)–(16), we can state $w = w_1 + w_2 + w_3$ with $w_i = h_{i,1}(v_i, v_i) + h_{i,2}(v_j, -v_j)$ for $i, j \neq 2, i \neq j$, and $w_2 = (h_{2,1}v_2, h_{2,2}v_2)$, where $h_{i,j} \in \mathbb{C}$. Based on those considerations, we estimate the Hermitian product $\bar{w}A(\xi)w$ as

$$\begin{aligned} \bar{w}A(\xi)w &= \bar{w}A_1(\xi)w + \bar{w}A_2(\xi)w + \bar{w}A_3(\xi)w \\ &\geq \bar{w}A_1(\xi)w + \bar{w}A_2(\xi)w = \sum_i^3 (\bar{w}_i A_1(\xi)w_i + \bar{w}_i A_2(\xi)w_i) \\ &= (|h_{11}|^2 + |h_{12}|^2 + |h_{21}|^2 + |h_{31}|^2 + |h_{32}|^2)|\xi|^{2\gamma} \\ &\quad + (|h_{11}|^2 + |h_{12}|^2 + |h_{22}|^2 + |h_{31}|^2 + |h_{32}|^2)(|\xi|^{2\beta} + 2) \\ &\quad - |\xi||w_1|^2 + |\xi||w_3|^2 \\ &= (|w_1|^2 + |h_{21}|^2 + |w_3|^2)|\xi|^{2\gamma} + (|w_1|^2 + |h_{22}|^2 + |w_3|^2)(|\xi|^{2\beta} + 2) \end{aligned}$$

$$\begin{aligned}
& -|\xi||w_1|^2 + |\xi||w_3|^2 \\
& = |w_1|^2(|\xi|^{2\gamma} + |\xi|^{2\beta} + 2 - |\xi|) + |w_3|^2(|\xi|^{2\gamma} + |\xi|^{2\beta} + 2 + |\xi|) \\
& \quad + |h_{21}|^2|\xi|^{2\gamma} + |h_{22}|^2(|\xi|^{2\beta} + 2) \\
& \geq |w|^2 \min\{|\xi|^{2\gamma}, |\xi|^{2\beta} + 2, |\xi|^{2\gamma} + |\xi|^{2\beta} + 2 - |\xi|\} \equiv |w|^2 T(\xi).
\end{aligned}$$

Now, letting $\delta = \min\{\gamma, \beta\} > \frac{1}{2}$, we have that

$$T(\xi) \geq \begin{cases} |\xi|^{2\gamma} & \text{if } |\xi| \leq 1, \\ C|\xi|^{2\delta} & \text{if } |\xi| > 1. \end{cases} \quad (17)$$

Therefore we get the following estimate for the eigenvalues of matrix $A(\xi)$:

$$\begin{cases} |\xi|^{2\gamma} < \min_{\lambda_i \in \sigma(A)} \lambda_i & \text{if } |\xi| \leq 1, \\ C|\xi|^{2\delta} < \min_{\lambda_i \in \sigma(A)} \lambda_i & \text{if } |\xi| > 1. \end{cases} \quad (18)$$

Now, we are going to prove the estimates for the expression $e^{-A(\xi)t}$. Next, we recall a fact about exponential Hermitian matrix with all positive eigenvalues.

Proposition 3.2. *Let M be a Hermitian matrix with all eigenvalues $\{\lambda_i\}_{i=1}^n$ positive. Then, for all $t \geq 0$,*

$$\|e^{-Mt}\| \leq e^{-(\min_i \lambda_i)t}. \quad (19)$$

Proof. Being M a Hermitian matrix, it is diagonalizable. Let $BS = \{v_1, v_2, \dots, v_n\}$ be a base of the eigenvectors for M with eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Then BS is also a base of the eigenvectors for e^{-Mt} with eigenvalues $\{e^{-\lambda_1 t}, e^{-\lambda_2 t}, \dots, e^{-\lambda_n t}\}$. Let $w \in \mathbb{C}^n$, $w = \sum_{i=1}^n \alpha_i v_i$, with $\alpha_i \in \mathbb{C}$. Since all norms in \mathbb{C}^n are equivalent, without loss of generality, we take $\|w\| = \max_i |\alpha_i|$, which is the maximum norm with respect to the base BS . Here, $|\cdot|$ denotes the standard norm in \mathbb{C} . Then for $t \geq 0$, we have

$$\|e^{-Mt} w\| = \left\| \sum_{i=1}^n \alpha_i e^{-\lambda_i t} v_i \right\| \leq \|w\| \left(\max_{i=1, \dots, n} |e^{-\lambda_i t}| \right) = \|w\| e^{-(\min_i \lambda_i)t},$$

and this implies that

$$\|e^{-Mt}\| \leq e^{-(\min_i \lambda_i)t}. \quad \square$$

As from this line onwards, C will denote generic constants that may change from line to line and are independent of $\xi \in \mathbb{R}^3$.

Let $\delta = \min\{\gamma, \beta\} > \frac{1}{2}$. Therefore, by using (18) and (19), it follows that for $t \geq 0$ and $|\xi| \neq 0$, there exists $C = C(\gamma, \beta) > 0$ (independent of ξ) such that

$$\|e^{-tA(\xi)}\| \leq e^{-(\min_i \lambda_i(\xi))t} \leq \begin{cases} e^{-|\xi|^{2\gamma}t} & \text{if } |\xi| \leq 1, \\ e^{-C|\xi|^{2\delta}t} & \text{if } |\xi| > 1. \end{cases} \quad (20)$$

Let $y = (u, \omega)$. For a field $v \in \mathbb{R}^3$, we will consider the notation $v \otimes y = (v \otimes u, v \otimes \omega)$, where \otimes denotes the tensorial product in \mathbb{R}^3 . Furthermore, we will denote, abusively $\mathbb{P}(v \otimes y) = (\mathbb{P}(v \otimes u), v \otimes \omega)$. Hence, we define

$$\widehat{\mathbb{P}(v \otimes y)} = (\widehat{\mathbb{P}(v \otimes u)}, \widehat{v \otimes \omega}) = (\widehat{\mathbb{P}(\xi)}(\widehat{v \otimes u}), \widehat{v \otimes \omega}).$$

We are now in a position to introduce the notion of solution for the system (7)–(9), with the help of Duhamel's principle.

Definition 3.3. Let $0 < T \leq \infty$. A mild solution to system (7)–(9) with initial data $y_0 = (u_0, \omega_0)$, $\operatorname{div} u_0 = 0$, in PM^a -space, is a time-dependent distribution $y = (u, \omega) \in F_{a,T}$ such that

$$\hat{y}(t, \xi) = (\hat{u}, \hat{\omega}) = e^{-tA(\xi)} \hat{y}_0 - \int_0^t e^{-(t-s)A(\xi)} i \xi \widehat{\mathbb{P}}(\xi) (\widehat{u \otimes y})(s, \xi) ds, \quad (21)$$

for all $0 < t < T$ and which satisfies

$$\operatorname{div} u = 0 \quad \text{and} \quad y(t) \rightarrow y_0 \quad \text{when } t \rightarrow 0^+,$$

in the distributional sense.

Our main results are the following.

Theorem 3.4. Let $a = 4 - 2\gamma$ with $\frac{1}{2} < \gamma = \min\{\beta, \gamma\} < \frac{5}{4}$. Let u_0 and $\omega_0 \in PM^a$ with $\operatorname{div} u_0 = 0$. There exists $\varepsilon > 0$ such that if $\|u_0\|_{PM^a}, \|\omega_0\|_{PM^a} < \varepsilon$, then the initial value problem (7)–(9) has a global mild solution $y = (u, \omega) \in F_a$. Moreover, if $\|y\|_{F_a} \leq 2\varepsilon$ then the solution is unique in this class of functional spaces.

Furthermore, if we assume $y_0 \in PM^a \cap PM^p$, with $2\gamma - 1 < p < 3$, then there exists $\varepsilon_p > 0$, $0 < \varepsilon_p \leq \varepsilon$, such that if $\|u_0\|_{PM^a}, \|\omega_0\|_{PM^a} < \varepsilon_p$, then the previous solution $y(t) = (u(t), \omega(t))$ verifies the additional property:

$$(u, \omega) \in F_p = BC((0, \infty); PM^p).$$

Theorem 3.5 (Regularization). Under the assumptions of the previous theorem, for any $a < q < 3$, there exists $0 < \varepsilon_q < \varepsilon$ such that if $\|u_0\|_{PM^a}, \|\omega_0\|_{PM^a} < \varepsilon_q$, then the solution $y = (u, \omega)$ given by Theorem 3.4 belongs to F_q and satisfies $\|y\|_{F_q} \leq 2\varepsilon_q$.

Theorem 3.6. Let $\frac{1}{2} < \min\{\beta, \gamma\} < \frac{5}{4}$. Let $a = 4 - 2\min\{\beta, \gamma\}$ when $\gamma > \min\{\beta, \gamma\}$ or let $4 - 2\min\{\beta, \gamma\} < a < 3$. If the initial data $(u_0, \omega_0) \in PM^a$ with $\operatorname{div} u_0 = 0$, then there exists $0 < T < \infty$ such that the initial value problem (7)–(9) has a unique mild solution $y = (u, \omega) \in F_{a,T}$.

Remark 3.7 (Smoothness).

- Adapting the arguments of T. Kato in [3], we can prove that the solutions of Theorem 3.5 are C^∞ -smooth instantly and are solutions of system (7)–(9) for $t > 0$, in classical sense. However, we do not know if this property is true for solutions of Theorem 3.4 (it is possible that they are singular, see Section 3.3). The essential point is that, for $t > 0$, the solutions given by Theorem 3.5 lie in $PM^a \cap PM^q \subset L^r$ ($a = 4 - 2\gamma < q < 3$ and $\frac{3}{3-a} < r < \frac{3}{3-q}$) and this fact does not hold for the solutions of Theorem 3.4.
- An analogous version of the results of Theorem 3.5 can be obtained for the solutions given by Theorem 3.6 and, therefore, if the initial data is small enough, then the solutions are also C^∞ -smooth instantly and are solutions of system (7)–(9) when $0 < t < T$, in classical sense.

Remark 3.8. All theorems above remain true if we consider the system (1)–(4) with an external moment g and an external force f to be nonnull with suitable smallness conditions on the respective norms.

3.2. Proofs of the well-posedness theorems

Well-posedness theorems will be a consequence of the following lemma for generic Banach spaces:

Lemma 3.9. (See [5].) *Let X be a Banach space with norm $\|\cdot\|_X$, and $B: X \times X \rightarrow X$ be a continuous bilinear map, i.e., there exists $K > 0$ such that for all $x_1, x_2 \in X$ we have*

$$\|B(x_1, x_2)\|_X \leq K \|x_1\|_X \|x_2\|_X.$$

Given $0 < \varepsilon < \frac{1}{4K}$ and $y \in X$, $y \neq 0$, such that $\|y\|_X < \varepsilon$, there exists a solution $x \in X$ for the equation $x = y + B(x, x)$ such that $\|x\|_X \leq 2\varepsilon$. The solution x is unique in the ball $\bar{B}(0, 2\varepsilon)$. Moreover, the solution depends continuously on y in the following sense: If $\|\tilde{y}\|_X \leq \varepsilon$, $\tilde{x} = \tilde{y} + B(\tilde{x}, \tilde{x})$, and $\|\tilde{x}\|_X \leq 2\varepsilon$, then

$$\|x - \tilde{x}\|_X \leq \frac{1}{1 - 4K\varepsilon} \|y - \tilde{y}\|_X.$$

As a consequence of the above lemma, we need to verify the continuity of the bilinear terms in the integral form of the 3DGMP equation to obtain the well-posedness theorems for the solutions of the integral equation (21).

3.2.1. Proof of Theorems 3.4 and 3.6

We first show the continuity of the bilinear form $B(y_1, y_2)$, $y_1 = (u_1, \omega_1)$, $y_2 = (u_2, \omega_2)$ in the mild formulation (21) defined as

$$\widehat{B(y_1, y_2)} = - \int_0^t e^{-(t-s)A(\xi)} i\xi \cdot \widehat{\mathbb{P}}(\xi) (\widehat{u_1 \otimes y_2})(s, \xi) ds. \quad (22)$$

Lemma 3.10. *Let $\frac{1}{2} < \delta = \min\{\beta, \gamma\} < \frac{5}{4}$. Let $a = 4 - 2\gamma$ and $\gamma = \delta$. Then there exists a constant K_{F_a} such that*

$$\|B(y_1, y_2)\|_{F_a} \leq K_{F_a} \|y_1\|_{F_a} \|y_2\|_{F_a}, \quad (23)$$

for all $y_1 = (u_1, \omega_1)$ and $y_2 = (u_2, \omega_2) \in F_a$. Moreover, if $y_1 \in F_p$ and $y_2 \in F_a$ with $2\gamma - 1 < p < 3$, then

$$\|B(y_1, y_2)\|_{F_p} \leq K_{F_p} \|y_1\|_{F_p} \|y_2\|_{F_a}. \quad (24)$$

Furthermore, let $a = 4 - 2\delta$ when $\gamma > \delta$ or $4 - 2\delta < a < 3$. If $0 < T < \infty$, then

$$\|B(y_1, y_2)\|_{F_{a,T}} \leq K_{F_{a,T}} \|y_1\|_{F_{a,T}} \|y_2\|_{F_{a,T}}, \quad (25)$$

for all $y_1 = (u_1, \omega_1)$ and $y_2 = (u_2, \omega_2) \in F_{a,T}$, where $K_{F_{a,T}} = C \max(T^{\tau_1}, T^{\tau_2})$ for some $\tau_1, \tau_2 > 0$.

Proof. We will omit the proof of estimate (24) because we can get it of analogous form at inequality (23); therefore, we just need to prove the inequalities (23) and (25).

For this, since $\frac{3}{2} < a < 3$, using elementary properties of the Fourier transform and Proposition 2.2, we obtain

$$\begin{aligned} & |(\widehat{u_1 \otimes y_2})(s, \xi)| \\ & \leq C \left(\int_{\mathbb{R}^3} \frac{1}{|\xi - z|^a} \frac{1}{|z|^a} dz \|u_1\|_{F_a} \|u_2\|_{F_a} + \int_{\mathbb{R}^3} \frac{1}{|\xi - z|^a} \frac{1}{|z|^a} dz \|u_1\|_{F_a} \|\omega_2\|_{F_a} \right) \\ & \leq \frac{K_{F_a}}{|\xi|^{2a-3}} \|u_1\|_{F_a} \|y_2\|_{F_a}. \end{aligned} \quad (26)$$

Using the estimate (20), we have that

$$\begin{aligned} |\xi|^a |\widehat{B(y_1, y_2)}| & \leq K_{F_a} \int_0^t \|e^{-(t-s)A(\xi)}\| |\xi|^{a+1} \left(\frac{1}{|\xi|^{2a-3}} \right) ds \|u_1\|_{F_a} \|y_2\|_{F_a} \\ & \leq K_{F_a} \int_0^t \|e^{-(t-s)A(\xi)}\| |\xi|^{4-a} ds \|u_1\|_{F_a} \|y_2\|_{F_a}. \end{aligned}$$

Therefore,

$$\|\widehat{B(y_1, y_2)}\|_{F_a} \leq K_{F_a} \sup_{t \geq 0, \xi \in \mathbb{R}^3} (I(t, \xi)) \|u_1\|_{F_a} \|y_2\|_{F_a},$$

where

$$I(t, \xi) = \int_0^t \|e^{-(t-s)A(\xi)}\| |\xi|^{4-a} ds.$$

Let 1_Ω denote the characteristic function on Ω . In order to estimate I , we separate it in two parts $I = I_1 + I_2$, where $I_1 = I \cdot 1_{\{|\xi| \leq 1\}}$ and $I_2 = I \cdot 1_{\{|\xi| \geq 1\}}$. Now, we consider two cases: $a = 4 - 2\delta$ and $a > 4 - 2\delta$. In the estimates below, we will use the estimate (20) in the respective cases.

We recall that the constant C below denotes generic constants that may change from line to line.

First case ($a = 4 - 2\delta$)

(i) If $\gamma = \delta$, then

$$I(t, \xi) \leq \int_0^t e^{-C(t-s)|\xi|^{2\gamma}} |\xi|^{2\gamma} ds = C(1 - e^{-C|\xi|^{2\gamma}}) \leq C.$$

(ii) If $\gamma > \delta$, then

$$\begin{aligned}
I_1(t, \xi) &\leq \int_0^t 1_{\{|\xi| \leq 1\}} e^{-(t-s)|\xi|^{2\gamma}} \left| \xi(t-s)^{\frac{1}{2\gamma}} \right|^{2\delta} (t-s)^{-\frac{\delta}{\gamma}} ds \\
&\leq \sup_{s>0} (s^{2\delta} e^{-s^{2\gamma}}) \times \int_0^T (t-s)^{-\frac{\delta}{\gamma}} ds = CT^{1-\frac{\delta}{\gamma}} \quad \text{and} \\
I_2(t, \xi) &\leq \int_0^t 1_{\{|\xi| > 1\}} e^{-C(t-s)|\xi|^{2\delta}} |\xi|^{2\delta} ds = C(1 - e^{-C|\xi|^{2\gamma}}) \leq C.
\end{aligned}$$

Second case ($a > 4 - 2\delta$)

(i) If $\gamma > \delta$, then

$$\begin{aligned}
I_1(t, \xi) &\leq \int_0^t 1_{\{|\xi| \leq 1\}} e^{-(t-s)|\xi|^{2\gamma}} \left| \xi(t-s)^{\frac{1}{2\delta}} \right|^{4-a} (t-s)^{-\frac{4-a}{2\delta}} ds \\
&\leq \sup_{s>0} (s^{4-a} e^{-s^{2\gamma}}) \times T^{1-\frac{4-a}{2\delta}} = CT^{1-\frac{4-a}{2\delta}}, \\
I_2(t, \xi) &\leq \int_0^t 1_{\{|\xi| > 1\}} e^{-C(t-s)|\xi|^{2\delta}} \left| \xi(t-s)^{\frac{1}{2\delta}} \right|^{4-a} (t-s)^{-\frac{4-a}{2\delta}} ds = CT^{1-\frac{4-a}{2\delta}}.
\end{aligned}$$

(ii) If $\gamma = \delta$, then

$$I(t, \xi) \leq \int_0^t e^{-C(t-s)|\xi|^{2\gamma}} \left| \xi(t-s)^{\frac{1}{2\gamma}} \right|^{4-a} (t-s)^{-\frac{4-a}{2\gamma}} ds \leq CT^{1-\frac{4-a}{2\gamma}}. \quad \square$$

In order to use Lemma 3.9, to prove well-posedness theorems, we need to obtain estimates for the linear part of the integral equation (21). This is the contents of the next lemma.

Lemma 3.11. Let $0 < T \leq \infty$ and $0 < a < 3$. If $y_0 = (u_0, \omega_0) \in PM^a$ then

$$\sup_{0 < t < T} \|G(t)y_0\|_a \leq \|y_0\|_a.$$

Moreover,

$$G(t)y_0 \rightharpoonup y_0 \quad \text{when } t \rightarrow 0^+,$$

in the distribution sense.

Proof. By (20) note that $\|e^{-tA(\xi)}\| \leq 1$ for all $\xi \in \mathbb{R}^3$. Thus, by definition of PM^a -spaces, we have that

$$\begin{aligned}
\|G(t)y_0\|_a &\leq |\xi|^a |e^{-tA(\xi)} \hat{y}_0(\xi)| \leq |\xi|^a \|e^{-tA(\xi)}\| |\hat{y}_0(\xi)| \\
&\leq \|y_0\|_a.
\end{aligned}$$

Let us prove the continuity in distributional sense. First, note that $\|A(\xi)\| \leq C(|\xi|^{2\max\{\gamma, \beta, 1\}} + 1)$. Then, for $\varphi \in S(\mathbb{R}^3)$ we obtain

$$\begin{aligned} |\langle G(t)y_0 - y_0, \varphi \rangle| &= \left| \int (e^{-A(\xi)t} - I) \hat{y}_0 \hat{\varphi} d\xi \right| = \left| \int \left(\int_0^t -A(\xi) e^{-A(\xi)s} \hat{y}_0 ds \right) \hat{\varphi} d\xi \right| \\ &\leq t \|y_0\|_{PM^a} \left(\sup_{\xi \in \mathbb{R}^3} \sup_{s>0} \|e^{-A(\xi)s}\| \right) \left\| \frac{\hat{\varphi} \|A(\xi)\|}{|\xi|^a} \right\|_{L^1(\mathbb{R}^3)}. \end{aligned} \quad (27)$$

Using the inequality (20), we have

$$\sup_{\xi \in \mathbb{R}^3} \sup_{s>0} \|e^{-A(\xi)s}\| \leq 1.$$

Note that, if $R > 0$ and $a < 3$, then

$$\left\| \frac{1}{|\xi|^a} \right\|_{L^1(B(0,R))} < \infty.$$

As $\hat{\varphi} \in S(\mathbb{R}^3)$, we obtain

$$\left\| \frac{\hat{\varphi} \|A(\xi)\|}{|\xi|^a} \right\|_{L^1(\mathbb{R}^3)} \leq C \|\hat{\varphi} |\xi|^{2\max\{\gamma, \beta, 1\}-a} + \hat{\varphi} |\xi|^{-a}\|_{L^1(\mathbb{R}^3)} < \infty.$$

Finally, using the last inequality and (27), we get

$$|\langle G(t)y_0 - y_0, \varphi \rangle| \rightarrow 0, \quad \text{when } t \rightarrow 0^+. \quad \square$$

A direct application of Lemma 3.9 in F_a and $F_{a,T}$ spaces completes the proof of well-posedness of the integral equation (21). For this, note that in case of Theorem 3.4 we can take the initial data sufficiently small, in a way that the hypothesis of Lemma 3.9 is verified.

In case of Theorem 3.6, we have that

$$K_{F_{T,a}} = C \max(T^{\tau_1}, T^{\tau_2}) \rightarrow 0, \quad \text{when } T \rightarrow 0,$$

and therefore, we can take $T > 0$ sufficiently small such that

$$\|G(t)y_0\|_{F_{a,T}} K_{F_{a,T}} \leq \|y_0\|_a K_{F_{a,T}} < \frac{1}{4}.$$

On the other hand, we need to show that $B(y_1, y_2)(t) \rightarrow 0$ as $t \rightarrow 0^+$ in the distributional sense, but we omit the proof because this follows a similar form at the second part of the proof corresponding to Lemma 3.11.

The final part of the theorem, i.e., $y(t) = (u(t), \omega(t)) \in F_p$ for initial data $y_0 \in PM^p \cap PM^a$ with $2\gamma - 1 < p < 3$, can be proven as follows. Since the solution given by Lemma 3.9 is obtained by recursion:

$$\begin{aligned} y_1(t, x) &= G(t)y_0, \\ y_{k+1}(t, x) &= y_1(t, x) - B(y_k, y_k), \end{aligned}$$

where $k \in \mathbb{N}$, we can use Lemmas 3.10 and 3.11 in order to obtain

$$\sup_{t>0} \|y_1(t)\|_p \leq \|y_0\|_p$$

and

$$\sup_{t>0} \|y_{k+1}(t)\|_p \leq \|y_0\|_p + K_{Fp} \sup_{t>0} \|y_k(t)\|_a \sup_{t>0} \|y_k(t)\|_p.$$

Now, let us choose $0 < \varepsilon_p \leq \varepsilon$ such that $2K_p \varepsilon_p < 1$ and assume that $\|y_0\|_a < \varepsilon_p$. The first part of Theorem 3.4 shows that $\sup_{t>0} \|y_k(t)\|_a \leq 2\varepsilon_p$. Therefore, we can estimate

$$\sup_{t>0} \|y_{k+1}(t)\|_p \leq \|y_0\|_p + 2K_p \varepsilon_p \sup_{t>0} \|y_k(t)\|_p.$$

Let us denote $M_0 = \|y_0(t)\|_p$ and $M_k = \sup_{t>0} \|y_k(t)\|_p$, then the sequence $\{M_k\}$ satisfies

$$M_{k+1} \leq M_0 + 2K_p \varepsilon_p M_k.$$

Taking $R = 2K_p \varepsilon_p < 1$, we can write

$$M_k \leq (1 + R + R^2 + \cdots + R^k) M_0 \leq \frac{1}{1 - R} M_0,$$

and thus,

$$w_{k+1} = y_{k+1} - y_k = -B(y_k, y_k) + B(y_{k-1}, y_{k-1}) = -B(w_k, y_k) - B(y_{k-1}, w_k).$$

Finally, Lemma 3.10 implies

$$\sup_{t>0} \|w_{k+1}\|_p \leq 2 \sup_{t>0} \|y_k(t)\|_p \sup_{t>0} \|w_k(t)\|_a.$$

Since $\lim_{k \rightarrow \infty} \|w_k\|_{F_a} = \lim_{n \rightarrow \infty} \|y_{k+1} - y_k\|_{F_a} = 0$, then the sequence $\{y_k\}$ is a Cauchy sequence in $BC((0, \infty); PM^p)$ and thus, it converges to some $\tilde{y}(t, x) \in BC((0, \infty); PM^p)$. The uniqueness of limit in distributional sense gets the desired conclusion.

3.2.2. Proof of Theorem 3.5

Let $\alpha = \frac{q-a}{\gamma}$ with $a = 4 - 2\gamma < q < 3$. First, note that we can estimate $\sup_{t>0} t^{\frac{\alpha}{2}} \|\cdot\|_q$ of the linear part of the integral equation as

$$\begin{aligned} \|G(t)y_0\|_q &\leq |\xi|^q |e^{-tA(\xi)} \hat{y}_0(\xi)| \\ &\leq |\xi|^{q-a} e^{-Ct|\xi|^{2\gamma}} |\xi|^a |\hat{y}_0(\xi)| \\ &\leq t^{-\frac{q-a}{2\gamma}} \left(\sup_{s>0} s^{q-a} e^{-Cs^{2\gamma}} \right) \|y_0\|_a \\ &\leq Ct^{-\frac{\alpha}{2}} \|y_0\|_{4-2\gamma}. \end{aligned}$$

Now, in order to prove Theorem 3.5, applying Lemma 3.9, we just need to show the continuity of the bilinear form at F_q space. Since we already have the continuity of bilinear term (22) at F_a space, the continuity in the norm $\sup_{t>0} t^{\frac{\alpha}{2}} \|\cdot\|_q$ remains to be proven.

For this, using the inequalities (26) with $a = q$, we get

$$|(\widehat{u_1 \otimes y_2})(s, \xi)| \leq \frac{C}{|\xi|^{2q-3}} \|u_1(s)\|_q \|y_2(s)\|_q,$$

and then, we estimate

$$\begin{aligned}
|\xi|^q |\widehat{B(y_1, y_2)}| &\leq C \int_0^t e^{-(t-s)A(\xi)} \|\xi\|^{4-q} \|u_1(s)\|_q \|y_2(s)\|_q ds \\
&\leq CI(\xi, t) \sup_{t>0} t^{\frac{\alpha}{2}} \|u_1(t)\|_q \sup_{t>0} t^{\frac{\alpha}{2}} \|y_2(t)\|_q \\
&\leq C \sup_{t>0} t^{\frac{\alpha}{2}} \|u_1(t)\|_q \sup_{t>0} t^{\frac{\alpha}{2}} \|y_2(t)\|_q,
\end{aligned}$$

where

$$I(\xi, t) = \int_0^t \|e^{-(t-s)A(\xi)}\| |\xi|^{4-q} s^{-\alpha} ds.$$

Since $\gamma = \delta$ and $q > 4 - 2\gamma$, using the estimate (20), we conclude that

$$\begin{aligned}
I(\xi, t) &\leq \int_0^t e^{-C(t-s)|\xi|^{2\gamma}} |\xi|^{4-q} s^{-\alpha} ds \\
&\leq \sup_{s>0} (s^{4-q} e^{-Cs^{2\gamma}}) \int_0^t (t-s)^{-\frac{4-q}{2\gamma}} s^{-\alpha} ds \\
&= Ct^{-\frac{4-q}{2\gamma}-\alpha+1} = Ct^{-\frac{\alpha}{2}},
\end{aligned}$$

and the theorem proof is finished in this way.

3.3. Stationary solutions

The aim of this section is to prove the existence of stationary mild solutions for the system (7)–(9) in PM^a -spaces. In this section, we consider external forces $f(t, x) = f(x) \neq 0$ and $g(t, x) = g(x) \neq 0$.

Let $\gamma = \delta = \min\{\gamma, \beta\}$ and $y = (u, \omega)$ be a stationary solution of integral equation (21), i.e.,

$$\begin{aligned}
(u, \omega) &= e^{-tA(\xi)}(\hat{u}, \hat{\omega}) - \int_0^t e^{-(t-s)A(\xi)} i\xi \widehat{\mathbb{P}}(\xi)(\widehat{u \otimes y})(\xi) ds \\
&\quad + \int_0^t e^{-(t-s)A(\xi)} \widehat{\mathbb{P}}(\xi)(\hat{f}(\xi), \hat{g}(\xi)) ds,
\end{aligned}$$

for every $t > 0$.

Since $A(\xi)$ is a Hermitian matrix, we can decompose the matrix as

$$A(\xi) = M(\xi)J(\xi)M^{-1}(\xi),$$

where $J(\xi)$ denotes the diagonal matrix with diagonal $(\lambda_i)_{i=1}^6$ and $\sigma(A) = \{\lambda_i\}_{i=1}^6$. By considerations of Section 3.1 about matrix $A(\xi)$, we know that $\lambda > 0$ for all $\lambda \in \sigma(A)$, where $\sigma(A)$ denotes the set of eigenvalues of matrix A . By estimate (20) and letting $t \rightarrow \infty$, we obtain that

$$\lim_{t \rightarrow \infty} \|e^{-A(\xi)t}\| \leq \lim_{t \rightarrow \infty} e^{-C|\xi|^{2\delta}t} = 0, \quad \text{for all fixed } \xi \neq 0. \quad (28)$$

Note also that

$$e^{-tJ(\xi)} = \begin{bmatrix} e^{-t\lambda_1} & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & e^{-t\lambda_n} \end{bmatrix}, \quad (29)$$

where $\lambda_k \in \sigma(A)$ for all $k = 1, \dots, 6$. Moreover, using that $\lambda_i > 0$ for all $i = 1, \dots, 6$,

$$\lim_{t \rightarrow \infty} \left(\int_0^t e^{-sJ(\xi)} ds \right) = \lim_{t \rightarrow \infty} ((I - e^{-tJ(\xi)}) D_{i=1}^6(\lambda_i^{-1})) = D_{i=1}^6(\lambda_i^{-1}), \quad (30)$$

where $D_{i=1}^6(\lambda_i^{-1})$ denotes the 6×6 diagonal matrix with diagonal $(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \frac{1}{\lambda_4}, \frac{1}{\lambda_5}, \frac{1}{\lambda_6})$. The last integral means to integrate each element of matrix $e^{sJ(\xi)}$. We remember the notation $\widehat{\mathbb{P}}(\xi)(\hat{f}, \hat{g}) = (\widehat{\mathbb{P}}(\xi)\hat{f}, \hat{g})$. On the other hand, since we are assuming that (u, ω) is a stationary solution, then using (28)–(30), we have

$$\begin{aligned} & (\hat{u}, \hat{\omega}) \\ &= \lim_{t \rightarrow \infty} \left(e^{-tA(\xi)}(\hat{u}, \hat{\omega}) - M(\xi) \int_0^t e^{-(t-s)J(\xi)} ds M^{-1}(\xi) (i\xi \widehat{\mathbb{P}}(\xi)(\widehat{u \otimes y}) - \widehat{\mathbb{P}}(\xi)(\hat{f}, \hat{g})) \right) \\ &= - \lim_{t \rightarrow \infty} M((I - e^{-tJ(\xi)}) D_{i=1}^6(\lambda_i^{-1})) M^{-1}(i\xi \widehat{\mathbb{P}}(\xi)(\widehat{u \otimes y}) - \widehat{\mathbb{P}}(\xi)(\hat{f}, \hat{g})) \\ &= -M(\xi) D_{i=1}^6(\lambda_i^{-1}) M^{-1}(\xi) (i\xi \widehat{\mathbb{P}}(\xi)(\widehat{u \otimes y}) - \widehat{\mathbb{P}}(\xi)(\hat{f}, \hat{g})) \\ &= -M(\xi) \int_0^\infty e^{-sJ(\xi)} ds M^{-1}(\xi) (i\xi \widehat{\mathbb{P}}(\xi)(\widehat{u \otimes y}) - \widehat{\mathbb{P}}(\xi)(\hat{f}, \hat{g})) \\ &= - \int_0^\infty e^{-sA(\xi)} i\xi \widehat{\mathbb{P}}(\xi)(\widehat{u \otimes y})(\xi) ds + \int_0^\infty e^{-sA(\xi)} \widehat{\mathbb{P}}(\xi)(\hat{f}, \hat{g}) ds. \end{aligned} \quad (31)$$

In short, we have proven that if (u, ω) is a stationary solution of integral equation (21), then (u, ω) satisfies the integral equation (31). Let us remark that by proceeding in a contrary order, we can obtain the equivalence.

Therefore, in order to prove the existence and uniqueness of the stationary solution for the system (7)–(9), it is sufficient to study the existence and uniqueness of the integral equation (31). For the last, we have the following theorem.

Theorem 3.12. *Let $\gamma = \delta = \min\{\beta, \gamma\}$ and $a = 4 - 2\gamma$ with $\frac{1}{2} < \gamma < \frac{5}{4}$. Let $(f, g) \in PM^{a-2\gamma}$. There exist $\eta > 0$ and $\varepsilon > 0$ such that if $\|(f, g)\|_{a-2\gamma} < \eta$, then the system (7)–(9) has a stationary mild solution $y = (u, \omega) \in PM^a$. Moreover, (u, ω) is the unique solution which satisfies $\|(u, \omega)\|_a \leq 2\varepsilon$.*

Proof. Note that all the estimates for the bilinear form (22) in $BC((0, \infty); PM^a)$ also hold for the bilinear term in (31)

$$- \int_0^\infty e^{-sA(\xi)} i\xi \widehat{\mathbb{P}}(\xi)(\widehat{u \otimes y})(\xi) ds.$$

Now, in order to get the proof by applying Lemma 3.9, we just need to prove the following lemma.

Lemma 3.13. *Under the assumptions of Theorem 3.12, let $H \in S'$, such that $\widehat{H}(\xi) = \int_0^\infty e^{-sA(\xi)} \widehat{\mathbb{P}}(\xi)(\hat{f}, \hat{g}) ds$. Then,*

$$\|H\|_a \leq C \|(f, g)\|_{a-2\gamma}.$$

Proof.

$$\begin{aligned} |\xi|^a |\widehat{H}(\xi)| &\leq \int_0^\infty \|e^{-sA(\xi)}\| |\xi|^{a-(a-2\gamma)} ds \|(f, g)\|_{a-2\gamma} \\ &\leq C \int_0^\infty e^{-sC|\xi|^{2\gamma}} |\xi|^{2\gamma} ds \|(f, g)\|_{a-2\gamma} \\ &= \frac{1}{C} \|(f, g)\|_{a-2\gamma}. \quad \square \end{aligned}$$

Remark 3.14 (Singular solutions).

- Theorem 3.12 shows that the unique small stationary mild solution of system (1)–(4), with $f = 0$ and $g = 0$ is the null solution.
- We can consider versions of Theorems 3.4 and 3.5 with $f = f(t, x)$ and $g = g(t, x)$ as non-nulls (see Remark 3.8). Theorem 3.12 assures the existence of singular solutions to system (1)–(4). Indeed, it is sufficient to take convenient $(f, g) \in PM^{a-2\gamma}$ which is singular. Therefore, in the view point of singular solutions, the last observation shows the importance of choosing the space F_a .

4. Decay estimates

The aim of this section is to prove some results of decaying when we take more regular initial data, and particularly we state that the solutions vanish at the infinity for some initial data.

Theorem 4.1. *Let us consider the assumptions of Theorem 3.4, let $a = 4 - 2\gamma$, $2\gamma - 1 < p < 3$ and suppose that $y_0 \in PM^p \cap PM^a$. Given $p < r \leq a$ satisfying $\frac{r-p}{2\gamma} < \frac{a-p}{2\gamma} = \frac{\alpha}{2}$, then the solution provided by Theorem 3.4 satisfies*

$$t^{\frac{r-p}{2\gamma}} y(t) \in BC((0, \infty); PM^r).$$

Corollary 4.2. *Let $E = \overline{PM^p \cap PM^a}$ be the closure of $PM^p \cap PM^a$ in PM^a . If $y_0 = (u_0, \omega_0) \in E$, then the corresponding solution given by Theorem 4.1 satisfies*

$$\lim_{t \rightarrow \infty} \|u(t)\|_a = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \|\omega(t)\|_a = 0.$$

Remark 4.3. The previous corollary proves that mild solutions with initial data in the $E = \overline{PM^p \cap PM^a}$ space have a simple long-time diffusive behavior since all the solutions decay to 0 as $t \rightarrow \infty$.

One of the key points in this argument is that the intersection $PM^p \cap PM^a$ is dense in E . However, this property is not true for PM^a -spaces. Thus, mild solutions with initial data that lie in $PM^a \cap E^c$ have a more complex long-time diffusive behavior. A class of initial data with such regularity is the set of functions of type $\frac{C}{|x|^{3-a}}$. Here E^c denotes the complement of E .

4.1. Proof of Theorem 4.1

Let $y_0 = (u_0, \omega_0)$ and $a = 4 - 2\gamma$. Proceeding analogously as in the proof of Theorem 3.5, we get

$$\sup_{t>0} t^{\frac{r-p}{2\gamma}} \|G(t)y_0\|_r \leq C \|y_0\|_p.$$

Now, by second part of Theorem 3.4, we already know that if $y_0 \in PM^p \cap PM^a$ then the solution $y(t) = (u(t), \omega(t))$ satisfies

$$\sup_{t>0} \|y(t)\|_p < \infty.$$

Thus, in order to conclude the proof of Theorem 4.1, we need a result where we estimate the norm $\sup_{t>0} t^{\frac{r-p}{2\gamma}} \|\cdot\|_r$ of bilinear form, using the norm $\|\cdot\|_{F_a} + \sup_{t>0} \|\cdot\|_p$ of y , where $\|\cdot\|_{F_a} = \sup_{t>0} \|\cdot\|_a$. For this, we prove the following lemma.

Lemma 4.4. *Let $y_1(t) = (u_1(t), \omega_1(t)) \in F_p$ and $y_2(t) = (u_2(t), \omega_2(t)) \in F_{q,a}$. Under the assumptions of Theorem 4.1, we have*

$$\sup_{t>0} t^{\frac{r-p}{2\gamma}} \|B(y_1, y_2)\|_r \leq C \|u_1\|_{F_p} \|y_2\|_{F_{q,a}}.$$

Proof. Firstly, we remind the reader that we are in the case $a = 4 - 2\gamma$, $\alpha = \frac{q-a}{\gamma}$ and that

$$\|\cdot\|_{F_{q,a}} = \sup_{t>0} \|\cdot\|_a + t^{\frac{\alpha}{2}} \|\cdot\|_q.$$

Now, working analogously as in the proof of estimate (26), we obtain

$$|(\widehat{u_1 \otimes y_2})(s, \xi)| \leq \frac{C}{|\xi|^{p+q-3}} \|u_1(s)\|_p \|y_2(s)\|_q.$$

Note that $0 < k = \frac{r+4-q-p}{2\gamma} = \frac{r-p}{2\gamma} - \frac{\alpha}{2} + 1 < 1$. Therefore, we estimate

$$\begin{aligned} |\xi|^r |\widehat{B(y_1, y_2)}(\xi, t)| &\leq C \int_0^t e^{-(t-s)|\xi|^{2\gamma}} |\xi|^{r+1+3-q-p} s^{-\frac{\alpha}{2}} s^{\frac{\alpha}{2}} \|u_1(s)\|_p \|y_2(s)\|_q ds \\ &\leq C \int_0^t e^{-C(t-s)|\xi|^{2\gamma}} |\xi|^{2\gamma k} s^{-\frac{\alpha}{2}} ds \|u_1\|_{F_p} \|y_2\|_{F_{q,a}} \\ &\leq C \int_0^t (t-s)^{-k} s^{-\frac{\alpha}{2}} ds \sup_{s>0} (s^{2\gamma k} e^{-s^{2\gamma}}) \|u_1\|_{F_p} \|y_2\|_{F_{q,a}} \\ &\leq C t^{1-\frac{\alpha}{2}-k} \|u_1\|_{F_p} \|y_2\|_{F_{q,a}} = C t^{-\frac{r-p}{2\gamma}} \|u_1\|_{F_p} \|y_2\|_{F_{q,a}}. \end{aligned}$$

Taking the supreme in \mathbb{R}^3 , we conclude that

$$t^{\frac{r-p}{2\gamma}} \|B(y_1, y_2)\|_{PM^r} \leq C \|u_1\|_{F_p} \|y_2\|_{F_q}. \quad \square$$

4.2. Proof of Corollary 4.2

We recall that $\delta = \min\{\beta, \gamma\}$ and $a = 4 - 2\gamma$. Since $a < q < 3$ and $\frac{1}{2} < \gamma = \delta < \frac{5}{4}$, we have that $2a - q > a = 4 - 2\gamma > 2\gamma - 1$. Thus, we can choose p such that

$$2\gamma - 1 < p < 2a - q < a \quad \text{and} \quad a - p > q - a.$$

Applying Theorem 4.1 with $r = a$, we get that for any $y_{0,k} = (u_{0,k}, \omega_{0,k}) \in PM^a \cap PM^p$,

$$\lim_{t \rightarrow \infty} \|y_k(t)\|_a = 0.$$

We denote by BC_0 the closed subspace of BC consisting of functions that vanish as $t \rightarrow \infty$.

Given $y_0 = (u_0, \omega_0) \in \overline{PM^a \cap PM^p}$, by density, it can be approximated in PM^a -norm by functions $y_{0,k} \in PM^a \cap PM^p$. Taking into account the continuity with respect to the initial data contained in Theorems 3.4 and 3.5, we deduce that $y_k(t)$ converges in $BC((0, \infty); PM^a)$ towards $y(t)$. Since the corresponding solutions y_k belong to $BC_0((0, \infty); PM^a)$, it follows by closedness of the subspace BC_0 that

$$\lim_{t \rightarrow \infty} \|y(t)\|_a = 0,$$

concluding the proof of the corollary.

5. Asymptotic stability

We analyze the large time behavior of solutions of Section 3. In short, we will show that small perturbations of the linear equation (10)–(11) are negligible for large times. We make precise this statement in the following results.

Theorem 5.1. Assume that $y(t) = (u(t), \omega(t))$ and $z(t) = (\tilde{u}(t), \tilde{\omega}(t))$ are solutions of system (7)–(9) given by Theorem 3.4 corresponding to initial conditions $y_0 = (u_0, \omega_0)$ and $z_0 = (\tilde{u}_0, \tilde{\omega}_0) \in PM^a$, respectively, satisfying

$$\lim_{t \rightarrow \infty} \|G(t)(y_0 - z_0)\|_a = 0,$$

then

$$\lim_{t \rightarrow \infty} \|y(t) - z(t)\|_a = 0$$

holds.

Moreover, if we assume $y(t)$ and $z(t)$ are solutions of system (7)–(9) given by Theorem 3.5 corresponding to initial conditions y_0 and $z_0 \in PM^a$ satisfying

$$\lim_{t \rightarrow \infty} t^{\frac{\alpha}{2}} \|G(t)(y_0 - z_0)\|_q = 0,$$

then

$$\lim_{t \rightarrow \infty} t^{\frac{\alpha}{2}} \|y(t) - z(t)\|_q = 0$$

holds.

Remark 5.2. Let us stress that the previous results applied to the particular case in which one solution is stationary (see Theorems 3.12 and 3.4) imply the existence of a basin of attraction for each stationary solution. This basin of attraction is characterized completely by the fact that the linear part of the solutions has to match for the large times. This complicated dynamical picture was already shown for the Navier–Stokes system in 3D in [1] recently.

Remark 5.3.

- Under the hypotheses of Theorem 5.1, the asymptotic stability can also be obtained in the norm $\|\cdot\|_p$ by assuming $\lim_{t \rightarrow \infty} \|G(t)(y_0 - z_0)\|_p = 0$, where $2\gamma - 1 < p < a = 4 - 2\gamma$.
- Let $2\gamma - 1 < p < a$ and $z_0 = 0$. If $\lim_{t \rightarrow \infty} \|G(t)y_0\|_p = 0$ (example: $y_0 \in \overline{PM^p} \cap \overline{PM^a}$ or in particular $C_c^\infty(\mathbb{R}^3)$), a consequence of the previous theorem is that

$$\lim_{t \rightarrow \infty} \|y(t)\|_p = 0,$$

a decay that does not follow from Theorem 4.1.

5.1. Proof of Theorem 5.1

By subtracting the integral equation in (21) for $y(t)$ from the analogous expression for $z(t)$ and taking the norm $\|\cdot\|_a$ of the resulting equation we obtain

$$\begin{aligned} \|y(t) - z(t)\|_a &\leq \|G(t)(y_0 - z_0)\|_a + K_{F_a}(\|y\|_{F_a} + \|z\|_{F_a})I_1(t) \\ &\quad + K_{F_a}(\|y\|_{F_a} + \|z\|_{F_a})I_2(t), \end{aligned} \quad (32)$$

where

$$I_1(t) = \sup_{\xi \in \mathbb{R}^3} \int_0^{\rho t} |\xi|^{2\gamma} e^{-C(t-s)|\xi|^{2\gamma}} \|y(s) - z(s)\|_a ds$$

and

$$I_2(t) = \sup_{\xi \in \mathbb{R}^3} \int_{\rho t}^t |\xi|^{2\gamma} e^{-(t-s)|\xi|^{2\gamma}} \|y(s) - z(s)\|_a ds,$$

for a small constant $\rho > 0$ that will be chosen later.

Changing variables $s = tz$ in $I_1(t)$ and noting that

$$\sup_{\xi \in \mathbb{R}^3} (t-s)|\xi|^{2\gamma} e^{-(t-s)|\xi|^{2\gamma}} \leq \sup_{s>0} s e^s < \infty,$$

we estimate

$$\begin{aligned} I_1(t) &\leq C \sup_{\xi \in \mathbb{R}^3} \int_0^{\rho t} (t-s)^{-1} \|y(s) - z(s)\|_a ds \\ &\leq C \int_0^\rho (1-s)^{-1} \|y(ts) - z(ts)\|_a ds. \end{aligned} \quad (33)$$

We deal $I_2(t)$ directly by

$$\begin{aligned} I_2(t) &\leq \left(\sup_{\xi \in \mathbb{R}^3} \int_{\rho t}^t |\xi|^{2\gamma} e^{-(t-s)|\xi|^{2\gamma}} ds \right) \left(\sup_{\rho t \leq s \leq t} \|y(s) - z(s)\|_a \right) \\ &= \left(\sup_{\xi \in \mathbb{R}^3} (1 - e^{-t(1-\rho)|\xi|^{2\gamma}}) \right) \sup_{\rho t \leq s \leq t} \|y(s) - z(s)\|_a \\ &= \sup_{\rho t \leq s \leq t} \|y(s) - z(s)\|_a, \end{aligned} \quad (34)$$

since $\sup_{\xi \in \mathbb{R}^3} (1 - e^{-t(1-\rho)|\xi|^{2\gamma}}) = 1$. We remember that the solutions of Theorem 3.4 satisfy

$$\|y\|_{F_a} \leq 2\varepsilon \quad \text{and} \quad \|z\|_{F_a} \leq 2\varepsilon, \quad (35)$$

therefore, by last inequalities (32)–(35), we get

$$\begin{aligned} \|y(t) - z(t)\|_a &\leq \|G(t)(y_0 - z_0)\|_a + 4C\varepsilon K_{F_a} \int_0^\rho (1-s)^{-1} \|y(ts) - z(ts)\|_a ds \\ &\quad + 4\varepsilon K_{F_a} \sup_{\rho t \leq s \leq t} \|y(s) - z(s)\|_a, \end{aligned} \quad (36)$$

for all $t > 0$.

Now, let us define

$$\Gamma = \lim_{t \rightarrow \infty} \sup_{t \geq k} \|y(t) - z(t)\|_a = \lim_{k \in \mathbb{N}, k \rightarrow \infty} \sup_{t \geq k} \|y(t) - z(t)\|_a.$$

We will show that $\Gamma = 0$. The inequality

$$\sup_{t \geq k} \int_0^\rho (1-s)^{-1} \|y(ts) - z(ts)\|_a ds \leq \int_0^\rho (1-s)^{-1} \sup_{t \geq k} \|y(ts) - z(ts)\|_a ds$$

implies that

$$\lim_{t \rightarrow \infty} \sup_{t \geq k} \int_0^\rho (1-s)^{-1} \|y(ts) - z(ts)\|_a ds \leq \Gamma \log\left(\frac{1}{1-\rho}\right).$$

Since for all k ,

$$\sup_{t \geq k} \sup_{\rho t \leq s \leq t} \|y(s) - z(s)\|_a \leq \sup_{\rho k \leq s < \infty} \|y(s) - z(s)\|_a,$$

we obtain

$$\lim_{t \rightarrow \infty} \sup_{\rho t \leq s \leq t} \|y(s) - z(s)\|_a \leq \Gamma.$$

Computing $\limsup_{t \rightarrow \infty}$ in (36) and using the last inequalities, we have

$$\Gamma \leq 4\varepsilon K_{F_a} \left(C \log\left(\frac{1}{1-\rho}\right) + 1 \right) \Gamma.$$

If we take $\rho > 0$ sufficiently small, since $4\varepsilon K_{F_a} < 1$ and Γ is nonnegative, then $\Gamma = 0$.

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